Supplemental Material

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A Preliminaries

We introduced the product Lie group $\mathcal{G} := \mathrm{SE}_3 \times \mathbb{R}^6 \times (0,1)^{|\Omega|}$ in equation (3). The identity element of the Lie group is $\mathrm{Id} = (\mathbb{1}_4, \mathbf{0}_6, \frac{\mathbf{1}}{\mathbf{2}}_{|\Omega|})$, where $\mathbb{1}_4$ denotes the 4×4 identity matrix. The Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ is the product Lie algebra $\mathfrak{g} := \mathfrak{se}_3 \times \mathbb{R}^6 \times \mathbb{R}^{|\Omega|}$, where $\mathfrak{se}_3$ is given through

$$\mathfrak{se}_3 := \left\{ \begin{pmatrix} [\omega]_\times & t \\ 0 & 1 \end{pmatrix} \middle| \omega \in \mathbb{R}^3, t \in \mathbb{R}^3 \right\}. \tag{A.1}$$

$[\omega]_\times$ denotes as usual the operator that maps a vector $\omega \in \mathbb{R}^3$ into the space of skew-symmetric 3×3 matrices, denoted by $\mathfrak{so}_3$. We will also write $\mathrm{mat}_{\mathfrak{so}}$ for this operation. It is given through

$$\mathrm{mat}_{\mathfrak{so}}(\omega), [\omega]_\times : \mathbb{R}^3 \to \mathfrak{so}_3, \quad \mathrm{mat}_{\mathfrak{so}}(\omega) := [\omega]_\times := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{A.2}$$

To be consistent with the usual inner product, i.e. $\langle \omega, \omega \rangle \overset{!}{=} \mathrm{tr}([\omega]_\times [\omega]_\times^\top)$ we multiply the factor $\sqrt{2}$. The inverse operator of $\mathrm{mat}_{\mathfrak{so}}$ is $\mathrm{vec}_{\mathfrak{so}} := (\mathrm{mat}_{\mathfrak{so}})^{-1} : \mathfrak{so}_3 \to \mathbb{R}^3$. Given a vector $\omega \in \mathbb{R}^6$ we can map it onto an element in $\mathfrak{se}_3$ by using the operator $\mathrm{mat}_{\mathfrak{se}} : \mathbb{R}^6 \to \mathfrak{se}_3$ that is defined through

$$\mathrm{mat}_{\mathfrak{se}}(\omega) := \begin{pmatrix} \mathrm{mat}_{\mathfrak{so}}(\omega_{1:3}) & \omega_{4:6} \\ 0 & 1 \end{pmatrix}. \tag{A.3}$$

Here, $\omega_{i:j}$ denotes the vector with the elements from i to j in ω. The inverse operation is again denoted by $\mathrm{vec}_{\mathfrak{se}} = (\mathrm{mat}_{\mathfrak{se}})^{-1} : \mathfrak{se}_3 \to \mathbb{R}^6$. Finally, we define for the Lie algebra $\mathfrak{g}$ the operation $\mathrm{vec}_{\mathfrak{g}}$. For $\eta_1 \in \mathbb{R}^6$, $\eta_2 \in \mathbb{R}^6$ and $\eta_3 \in \mathbb{R}^{|\Omega|}$ it is given through

$$\mathrm{mat}_{\mathfrak{g}}(\eta) = \mathrm{mat}_{\mathfrak{g}}((\eta_1, \eta_2, \eta_3)) := (\mathrm{mat}_{\mathfrak{se}}(\eta_1), \eta_2, \eta_3) \in \mathfrak{g}\,. \tag{A.4}$$

The inverse operation is denote by $\mathrm{vec}_{\mathfrak{g}} := \mathrm{mat}_{\mathfrak{g}}^{-1} : \mathfrak{g} \to \mathbb{R}^{12+|\Omega|}$.

The corresponding Riemannian metric is given for $\eta^i = (\eta^i_E, \eta^i_v, \eta^i_{d_i}) \in \mathfrak{g}$, $i = 1, 2$ through

$$\langle \eta^1, \eta^2 \rangle_{\mathrm{Id}} = \langle \eta^1_E, \eta^2_E \rangle_{\mathfrak{se}_3} + \langle \eta^1_v, \eta^2_v \rangle + \langle \eta^1_{d_i}, \eta^2_{d_i} \rangle\,, \tag{A.5}$$

where $\langle \eta^1_E, \eta^2_E \rangle_{\mathfrak{se}_3} := \mathrm{tr}((\eta^1_E)^\top \eta^2_E)$ is the usual matrix inner product.

A.1 Vectorization of connection functions

Following [1, Section 5.2], we can vectorize the connection function ω of the Levi-Civita connection ∇ for *constant* η, $\xi \in \mathfrak{g}$ in the following way:

$$\mathrm{vec}_{\mathfrak{g}}(\omega_\eta \xi) = \mathrm{vec}_{\mathfrak{g}}(\omega(\eta, \xi)) = \mathrm{vec}_{\mathfrak{g}}(\nabla_\eta \xi) = \tilde{\Gamma}_{\mathrm{vec}_{\mathfrak{g}}(\xi)} \, \mathrm{vec}_{\mathfrak{g}}(\eta) \,, \tag{A.6}$$

where $\tilde{\Gamma}_x$ is the matrix whose (i, j) element is the real-valued function

$$(\tilde{\Gamma}_\gamma)_{i,j} := \sum_k (\gamma_k \Gamma^i_{jk}) \,, \tag{A.7}$$

and Γ^i_{jk} are the *Christoffel symbols* of the connection function ω for a vector $\gamma \in \mathbb{R}^{12\times|\Omega|}$. Similarly, permuting indices, we can define the adjoint matrix $\tilde{\Gamma}^*_\gamma$ whose (i, j)-th element is given by

$$(\tilde{\Gamma}^*_\gamma)_{i,j} := \sum_k (\gamma_k \Gamma^i_{kj}) \,. \tag{A.8}$$

This leads to the following equality:

$$\mathrm{vec}_{\mathfrak{g}}(\omega_\eta \xi) = \tilde{\Gamma}^*_{\mathrm{vec}_{\mathfrak{g}}(\eta)} \, \mathrm{vec}_{\mathfrak{g}}(\xi) \,. \tag{A.9}$$

If the expression ξ in (A.6) is *non-constant*, we obtain the following vectorization from [1, Eq. (5.7)], for the case of the Lie algebra $\mathfrak{se}_3$, i.e.

$$\begin{aligned} &\mathrm{vec}_{\mathfrak{se}}(\nabla_{\eta_x} \xi(x)) \\ =&\tilde{\Gamma}_{\mathrm{vec}_{\mathfrak{g}}(\xi(x))} \, \mathrm{vec}_{\mathfrak{se}}(\eta_x) + \mathbf{D} \, \mathrm{vec}_{\mathfrak{se}}(\xi(x))[\mathrm{vec}_{\mathfrak{se}}(\eta_x)] \\ =&\tilde{\Gamma}_{\mathrm{vec}_{\mathfrak{g}}(\xi(x))} \, \mathrm{vec}_{\mathfrak{se}}(\eta_x) + \sum_i (\eta_x)_i \, \mathrm{vec}_{\mathfrak{se}}(\mathbf{D}\xi(x))[E^i]) \\ =&\tilde{\Gamma}_{\mathrm{vec}_{\mathfrak{g}}(\xi(x))} \, \mathrm{vec}_{\mathfrak{se}}(\eta_x) + D \, \mathrm{vec}_{\mathfrak{se}}(\eta_x) \,, \end{aligned} \tag{A.10}$$

where the entries of the matrix $D \in \mathbb{R}^{6\times 6}$ can be computed as

$$(D)_{i,j} = (\mathrm{vec}_{\mathfrak{se}}(\mathbf{D}\xi(x)[E^j]))_i \,, \quad E^j = \mathrm{mat}_{\mathfrak{se}}(e^6_j) \,, \tag{A.11}$$

where e^6_j denotes the j-th unit vector in $\mathbb{R}^6$.

The adjoint operator $\mathrm{ad}_{\mathfrak{se}}(\mathrm{mat}_{\mathfrak{se}}(v))$ can be computed for a vector $v \in \mathbb{R}^6$ as follows

$$\begin{aligned} \mathrm{vec}_{\mathfrak{se}}\big(\mathrm{ad}_{\mathfrak{se}}(\mathrm{mat}_{\mathfrak{se}}(v))\eta\big) &= \mathrm{ad}^{\mathrm{vec}}_{\mathfrak{se}}(\mathrm{mat}_{\mathfrak{se}}(v)) \, \mathrm{vec}_{\mathfrak{se}}(\eta) \\ &:= \begin{pmatrix} \mathrm{mat}_{\mathfrak{so}}(v_{1:3}) & \mathbf{0}_{3\times 3} \\ \mathrm{mat}_{\mathfrak{so}}(v_{4:6}) & \mathrm{mat}_{\mathfrak{so}}(v_{1:3}) \end{pmatrix} \mathrm{vec}_{\mathfrak{se}}(\eta) \,, \end{aligned} \tag{A.12}$$

where $\mathrm{mat}_{\mathfrak{so}}(v_{1:3}) := (\mathrm{mat}_{\mathfrak{se}}(v))_{1:3,1:3}$. This directly follows from the definition of the adjoint as Lie bracket, i.e. $\mathrm{ad}_{\mathfrak{se}}(\xi)\eta := [\xi, \eta]$ where the Lie bracket $[\cdot, \cdot] : \mathfrak{se}_3 \times \mathfrak{se}_3 \to \mathfrak{se}_3$ is simply the matrix commutator on $\mathfrak{se}_3$. This leads to the following vectorization of the Lie bracket on $\mathfrak{g}$:

$$\operatorname{ad}_{\mathfrak{g}}^{\text{vec}}(\operatorname{mat}_{\mathfrak{g}}(v)) = \begin{pmatrix} \operatorname{ad}_{\mathfrak{se}}^{\text{vec}}(v_{1:6}) & \mathbf{0}_{6\times 6} & \mathbf{0}_{6\times|\Omega|} \\ \mathbf{0}_{6\times 6} & \mathbf{0}_{6\times 6} & \mathbf{0}_{6\times|\Omega|} \\ \mathbf{0}_{|\Omega|\times 6} & \mathbf{0}_{|\Omega|\times 6} & \mathbf{0}_{|\Omega|\times|\Omega|} \end{pmatrix}. \tag{A.13}$$

For the following calculations we require the *symmetry* property of the Levi-Civita connection ∇ which reads

$$\nabla_\eta \xi - \nabla_\xi \eta = [\xi, \eta] \tag{A.14}$$

where $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denotes the Lie bracket on $\mathfrak{g}$, cf. [1, p. 97].

A.2 Christoffel symbols on SE_3

The Christoffel symbols $\Gamma_{ij}^k, i, j, k \in \{1, \dots, 6\}$ for the Riemannian connection on SE_3 are given by

$$\begin{aligned} \Gamma_{12}^3 = \Gamma_{23}^1 = \Gamma_{31}^2 &= \tfrac{1}{2}\,, \\ \Gamma_{13}^2 = \Gamma_{21}^3 = \Gamma_{32}^1 &= -\tfrac{1}{2}\,, \\ \Gamma_{15}^6 = \Gamma_{26}^4 = \Gamma_{34}^5 &= 1\,, \\ \Gamma_{16}^5 = \Gamma_{24}^6 = \Gamma_{35}^4 &= -1\,. \end{aligned}$$

and zero otherwise.

B Proofs of Lemmas and Theorems

Proof of Lemma 1. The proof requires to compute the total time derivative of the necessary condition for the optimal state x^* which is given through

$$\mathbf{D}_1 \mathcal{V}(x^*, t) = \mathbf{0}. \tag{B.1}$$

The calculation of the time derivative of (B.1) is already given in [2, Eq. (26)–(37)] and results in the following evolution equation:

$$(x^*(t))^{-1}\dot{x}^*(t) = -\mathbf{D}_2\mathcal{H}(x^*(t), \mathbf{0}, t) - Z(x^*(t), t)^{-1} \circ (x^*)^{-1}(\mathbf{D}_1\mathcal{H}(x^*(t), \mathbf{0}, t)). \tag{B.2}$$

The derivative of the Hamiltonian regarding the second component is simply

$$\mathbf{D}_2\mathcal{H}(x^*, \mathbf{0}, t) = -f(x^*)\,, \tag{B.3}$$

such that the evolution equation for the optimal state x^* reads

$$(x^*(t))^{-1}\dot{x}^*(t) = f(x^*) - Z(x^*(t), t)^{-1} \circ (x^*)^{-1}(\mathbf{D}_1\mathcal{H}(x^*(t), \mathbf{0}, t)), \tag{B.4}$$

which is Lemma 1.

The calculation of the differential of the Hamiltonian

$$\mathbf{D}_1\mathcal{H}(x^*(t),\mathbf{0},t)$$

is a bit involved but can be calculated component-wise. We will use the short-hands $I := \left(\begin{smallmatrix}1&0&0&0\\0&1&0&0\end{smallmatrix}\right) \in \mathbb{R}^{2\times 4}$, $e = \left(\begin{smallmatrix}0&0&1&0\end{smallmatrix}\right) \in \mathbb{R}^{1\times 4}$, $g_z = \left(\begin{smallmatrix}\left(\begin{smallmatrix}z\\1\end{smallmatrix}\right)(d_i(z,t))^{-1}\\1\end{smallmatrix}\right) \in \mathbb{R}^4$ and $\kappa_z = eEg_z \in \mathbb{R}$. With these expressions we can write the function h as $h(z,t) := \kappa_z^{-1} IEg_z$.

We begin with the directional derivative of the Hamiltonian $\mathcal{H}(x,\mathbf{0},t)$ regarding the camera motion E for $x = (E,v,d_i)$ in a specific direction $E\eta \in T_E\,\mathrm{SE}_3$ which is

$$\begin{aligned}
\mathbf{D}_E\mathcal{H}(x,\mathbf{0},t)[E\eta] &= \sum_z \mathbf{D}_E\phi(\tfrac{1}{2}\|y_z - h_z(x,t)\|_{Q_z}^2)[E\eta] \\
&= \sum_z \beta(\tfrac{1}{2}\|y_z - h_z(x,t)\|_{Q_z}^2 + \nu)^{\beta-1}(y_z - h_z(x,t))^\top Q_z(-1)\mathbf{D}_E h_z(x,t)[E\eta] \\
&= \sum_z \beta(\cdots)^{\beta-1}(y_z - h_z(x,t))^\top Q_z(-1)\mathbf{D}_E\big(eEg_z IEg_z\big)[E\eta] \\
&= \sum_z \beta(\cdots)^{\beta-1}(-1)\,\mathrm{tr}\Big((y_z - h_z(x,t))^\top Q_z(\kappa_z^{-1}IE\eta g_z - \kappa_z^{-2}eE\eta g_z IEg_z)\Big) \\
&= \sum_z \beta(\cdots)^{\beta-1}(-1)\,\mathrm{tr}\Big((y_z - h_z(x,t))^\top Q_z\big(\kappa_z^{-1}I - \kappa_z^{-2}IEg_z e\big)E\eta g_z\Big) \\
&= \sum_z \beta(\cdots)^{\beta-1}(-1)\Big\langle g_z(y_z - h_z(x,t))^\top Q_z\big(\kappa_z^{-1}I - \kappa_z^{-2}IEg_z e\big)E\eta\Big\rangle \\
&= \sum_z \beta(\cdots)^{\beta-1}\Big\langle \Big(g_z(y_z - h_z(x,t))^\top Q_z\big(\kappa_z^{-2}IEg_z(e_3^4)^\top - \kappa_z^{-1}I\big)E\Big)^\top, \eta\Big\rangle .
\end{aligned}$$

From the definition of the Riemannian gradient on SE_3 follows that it can be computed by the orthogonal projection $\mathrm{Pr}: \mathbb{R}^{4\times 4} \to \mathfrak{se}_3$ which reads

$$A \mapsto \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\left(A\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&2\end{pmatrix} - A^\top\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\right). \tag{B.5}$$

The resulting Riemannian gradient of the Hamiltonian regarding E is

$$\begin{aligned}
\mathbf{D}_E\mathcal{H}(x,0,t) &= \sum_z \beta(\cdots)^{\beta-1}E\,\mathrm{Pr}\Big(\Big(g_z(y_z - h_z(x,t))^\top Q_z\big(\kappa_z^{-2}IEg_z e - \kappa_z^{-1}I\big)E\Big)^\top\Big) \\
&=: EG_E(x) \in T_E\,\mathrm{SE}_3\ .
\end{aligned} \tag{B.6}$$

The gradient of the Hamiltonian can be calculated component-wise, i.e. for each z in the image domain Ω separately. We will use the shorthand $g_z' := -\left(\begin{smallmatrix}\left(\begin{smallmatrix}z\\1\end{smallmatrix}\right)(d_i(z,t))^{-2}\\0\end{smallmatrix}\right)$ for the partial derivative $\frac{\partial}{\partial d_i(z,t)}g_z$.

Then the components of $\mathbf{D}_{d_i}\mathcal{H}(x,\mathbf{0},t)$ read

$$\frac{\partial}{\partial d_i(z,t)}\mathcal{H}(x,\mathbf{0},t) = \frac{\partial}{\partial d_i(z,t)}\sum_{z\in\Omega}\phi(\tfrac{1}{2}\|y_z - h_z(x,t)\|_{Q_z}^2)$$

$$
\begin{aligned}
&= \frac{\partial}{\partial d_i(z,t)} \left(\tfrac{1}{2}\|y_z - h_z(x,t)\|^2_{Q_z} + \nu\right)^\beta + \nu^\beta \\
&= \beta(\cdots)^{\beta-1}(y_z - h_z(x,t))^\top Q_z(-1)\frac{\partial}{\partial d_i(z,t)} h_z(x,t) \\
&= \beta(\cdots)^{\beta-1}(y_z - h_z(x,t))^\top Q_z(-1)\left(-\kappa_z^{-2} e E g'_z I E g_z + \kappa_z^{-1} I E g'_z\right) \\
&= \beta(\cdots)^{\beta-1}(y_z - h_z(x,t))^\top Q_z\left(\kappa_z^{-2} e E g'_z I E g_z - \kappa_z^{-1} I E g'_z\right) \\
&=: (G_{d_i}(x))_z \in \mathbb{R}\,. \qquad \text{(B.7)}
\end{aligned}
$$

By stacking these entries to a vector (given a fixed ordering of $z \in \Omega$, e.g. column-wise), we obtain the expression $G_{d_i}(x)$. Since the Hamiltonian does not depend on the vector v, the corresponding entries are zero such that we finally obtain the gradient of the Hamiltonian regarding $x = (E, v, d_i)$ which is

$$
\mathbf{D}_1\mathcal{H}(x, \mathbf{0}, t) = T_{\mathrm{Id}} L_x(G_E(x), \mathbf{0}_6, G_{d_i}(x)) \in T_x\mathcal{G}. \qquad \text{(B.8)}
$$

□

Proof of Lemma 2. Following Saccon et al. [2, Eq. (51)] the evolution equation of the operator $Z(x^*, t) : \mathfrak{g} \to \mathfrak{g}^*$ is given through

$$
\begin{aligned}
&\frac{d}{dt} Z(x^*, t) && \text{(B.9a)}\\
&\approx Z(x^*, t) \circ \omega_{(x^*)^{-1}\dot{x}^*} && \text{(B.9b)}\\
&\quad + Z(x^*, t) \circ \omega_{\overleftarrow{\mathbf{D}_2}\mathcal{H}(x^*,0,t)} && \text{(B.9c)}\\
&\quad + \omega^*_{(x^*)^{-1}\dot{x}^*} \circ Z(x^*, t) && \text{(B.9d)}\\
&\quad + \omega^*_{\overleftarrow{\mathbf{D}_2}\mathcal{H}(x^*,0,t)} \circ Z(x^*, t) && \text{(B.9e)}\\
&\quad + T_{\mathrm{Id}} L^*_{x^*} \circ \mathrm{Hess}_1\, \mathcal{H}(x^*, 0, t) \circ T_{\mathrm{Id}} L_{x^*} && \text{(B.9f)}\\
&\quad + T_{\mathrm{Id}} L^*_{x^*} \circ \mathbf{D}_2(\mathbf{D}_1\mathcal{H})(x^*, 0, t) \circ Z(x^*, t) && \text{(B.9g)}\\
&\quad + Z(x^*, t) \circ \mathbf{D}_1(\mathbf{D}_2\mathcal{H})(x^*, 0, t) \circ T_{\mathrm{Id}} L_{x^*} && \text{(B.9h)}\\
&\quad + Z(x^*, t) \circ \mathrm{Hess}_2\, \mathcal{H}(x^*, 0, t) \circ Z(x^*, t)\,. && \text{(B.9i)}
\end{aligned}
$$

Since $Z \in \mathfrak{g}^*$ and $\mathfrak{g}^*$ is a vector space we can represent $Z(x^*, t)$ as a matrix $K(t) \in \mathbb{R}^{(|\Omega|+12)\times(|\Omega|+12)}$ by evaluating Z for a specific $\eta \in \mathfrak{g}$ and vectorization, i.e.

$$
\mathrm{vec}_{\mathfrak{g}}(Z(x^*, t)(\eta)) = K(t)\, \mathrm{vec}_{\mathfrak{g}}(\eta)\,. \qquad \text{(B.10)}
$$

Similarly we can vectorize the full differential equation of Z to find the evolution equation of the operator $P(t)$ in Lemma 2.

With (B.10) it is obvious to see that the expression in (B.9a) can be vectorized as

$$
\mathrm{vec}_{\mathfrak{g}}\Big(\frac{d}{dt} Z(x^*, t)(\eta)\Big) = \dot{K}(t)\, \mathrm{vec}_{\mathfrak{g}}(\eta)\,. \qquad \text{(B.11)}
$$

Second, we consider the expressions in (B.9b) and (B.9c).

$$
\mathrm{vec}_{\mathfrak{g}}(Z(x^*, t) \circ \omega_{(x^*)^{-1}\dot{x}^*}\eta + Z(x^*, t) \circ \omega_{\overleftarrow{\mathbf{D}_2}\mathcal{H}(x^*,0,t)}\eta)
$$

$$
\begin{aligned}
&\overset{(B.10)}{=} K(t)\operatorname{vec}_{\mathfrak{g}}(\omega_{(x^*)^{-1}\dot{x}^*}\eta + \omega_{\overleftarrow{\mathbf{D}_2}\mathcal{H}(x^*,0,t)}\eta)\\
&\overset{(B.4)}{=} K(t)\operatorname{vec}_{\mathfrak{g}}(\nabla_{f(x^*)}\eta - \nabla_{Z(x^*,t)^{-1}\circ(x^*)^{-1}\mathbf{D}_1\mathcal{H}(x^*,\mathbf{0},t)}\eta\\
&\qquad + \nabla_\eta \mathbf{D}_2\mathcal{H}(x^*,0,t))\\
&\overset{(B.3)}{=} K(t)\operatorname{vec}_{\mathfrak{g}}(\nabla_{f(x^*)}\eta - \nabla_{Z(x^*,t)^{-1}\circ(x^*)^{-1}\mathbf{D}_1\mathcal{H}(x^*,\mathbf{0},t)}\eta\\
&\qquad - \nabla_\eta f(x^*))\\
&\overset{(A.14)}{=} K(t)\operatorname{vec}_{\mathfrak{g}}([f(x^*),\eta] - \nabla_{Z(x^*,t)^{-1}\circ(x^*)^{-1}\mathbf{D}_1\mathcal{H}(x^*,\mathbf{0},t)}\eta)\\
&\overset{(A.9)}{=} K(t)\big(\operatorname{vec}_{\mathfrak{g}}([f(x^*),\eta]) - \tilde{\varGamma}^*_{\operatorname{vec}_{\mathfrak{g}}(Z(x^*,t)^{-1}\circ(x^*)^{-1}\mathbf{D}_1\mathcal{H}(x^*,\mathbf{0},t))}\operatorname{vec}_{\mathfrak{g}}(\eta)\big)\\
&\overset{(B.10)}{=} K(t)\big(\operatorname{vec}_{\mathfrak{g}}([f(x^*),\eta]) - \tilde{\varGamma}^*_{K(t)^{-1}\operatorname{vec}_{\mathfrak{g}}((\dot{x})^{-1}\mathbf{D}_1\mathcal{H}(x^*,\mathbf{0},t))}\operatorname{vec}_{\mathfrak{g}}(\eta)\big)\\
&\overset{(A.13)}{=} K(t)\big(\operatorname{ad}^{\mathrm{vec}}_{\mathfrak{g}}(f(x^*)) - \tilde{\varGamma}^*_{K(t)^{-1}\operatorname{vec}_{\mathfrak{g}}((\dot{x})^{-1}\mathbf{D}_1\mathcal{H}(x^*,\mathbf{0},t))}\big)\operatorname{vec}_{\mathfrak{g}}(\eta)\\
&=: K(t)C_1(x^*,t)\operatorname{vec}_{\mathfrak{g}}(\eta)\,. && \text{(B.12)}
\end{aligned}
$$

By duality we find that the lines (B.9d) and (B.9e) can be represented as

$$
\operatorname{vec}_{\mathfrak{g}}\big(\omega^*_{(x^*)^{-1}\dot{x}^*}\circ Z(x^*,t)(\eta) + \omega^*_{\overleftarrow{\mathbf{D}_2}\mathcal{H}(x^*,0,t)}\circ Z(x^*,t)(\eta)\big) = C_1(x^*,t)^\top K(t)\eta. \tag{B.13}
$$

The vectorization of (B.9h) can be simply achieved as

$$
\begin{aligned}
&\operatorname{vec}_{\mathfrak{g}}\big(Z(x^*,t)\circ \mathbf{D}_1(\mathbf{D}_2\mathcal{H})(x^*,0,t)\circ T_{\mathrm{Id}}L_{x^*}\eta\big)\\
&\overset{(B.10)}{=} K(t)\operatorname{vec}_{\mathfrak{g}}\big(\mathbf{D}_1(\mathbf{D}_2\mathcal{H})(x^*,0,t)\circ x^*\eta\big)\\
&\quad = K(t)\operatorname{vec}_{\mathfrak{g}}\Big(-\mathbf{D}_1 f(x^*)[x^*\eta]\Big)\\
&\quad = -K(t)\operatorname{vec}_{\mathfrak{g}}(\operatorname{mat}_{\mathfrak{se}}((\eta_2,\mathbf{0}_6,\mathbf{0}_{|\varOmega|})))\\
&\quad = -K(t)\begin{pmatrix}\mathbf{0}_{6\times 6} & \mathbb{1}_6 & \mathbf{0}_{6\times|\varOmega|}\\ \mathbf{0}_{6\times 6} & \mathbf{0}_6 & \mathbf{0}_{6\times|\varOmega|}\\ \mathbf{0}_{|\varOmega|\times 6} & \mathbf{0}_{|\varOmega|\times 6} & \mathbf{0}_{|\varOmega|\times|\varOmega|}\end{pmatrix}\operatorname{vec}_{\mathfrak{g}}(\eta)\\
&\quad =: -K(t)C_2(x^*,t)\operatorname{vec}_{\mathfrak{g}}(\eta)\,,
\end{aligned}
$$

where f is the function in the paper defined in (6). Again, by duality follows

$$
\operatorname{vec}_{\mathfrak{g}}\big(T_{\mathrm{Id}}L^*_{x^*}\circ \mathbf{D}_2(\mathbf{D}_1\mathcal{H})(x^*,0,t)\circ Z(x^*,t)\circ\eta\big) = C_2(x^*,t)^\top K(t)\operatorname{vec}_{\mathfrak{g}}(\eta) \tag{B.14}
$$

for the expression in (B.9g). Setting $C(x^*,t) := C_2(x^*,t) - C_1(x^*,t)$ gives the matrix $C(x^*,t)$ in Lemma 2.

The expression in (B.9i) can be calculated as

$$
\begin{aligned}
&\operatorname{vec}_{\mathfrak{g}}\big(Z(x^*,t)\circ \operatorname{Hess}_2\mathcal{H}(x^*,0,t)\circ Z(x^*,t)\circ\eta\big) && \text{(B.15)}\\
&\overset{(B.10)}{=} K(t)\operatorname{vec}_{\mathfrak{g}}\big(\operatorname{Hess}_2\mathcal{H}(x^*,0,t)\circ Z(x^*,t)\big) && \text{(B.16)}\\
&\quad = -K(t)RK(t)\operatorname{vec}_{\mathfrak{se}}(\eta)\,, && \text{(B.17)}
\end{aligned}
$$

where R is the weighting matrix in the energy function in the original paper in (8).

It remains to calculate the matrix representation of the Hessian of the Hamiltonian in (B.9f), i.e.

$$\mathrm{vec}_{\mathfrak{g}}\big(T_{\mathrm{Id}}L^*_{x^*} \circ \mathrm{Hess}_1\,\mathcal{H}(x^*,0,t) \circ T_{\mathrm{Id}}L_{x^*} \circ \eta\big) =: H(x^*,t)\,\mathrm{vec}_{\mathfrak{g}}(\eta). \tag{B.18}$$

The single blocks of the Hessian of the Hamiltonian can be calculated again separately, i.e.

$$H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}.$$

Note that the entries that address the variable v are all zero since the Hamiltonian does not depend on v for $x = (E, v, d_i)$. Thus, the entries $H_{12}, H_{13}, H_{21}, H_{22}, H_{31}$ are all zero. As we consider the Riemannian Hessian regarding the symmetric Levi-Civita connection, it is sufficient to calculate H_{11}, H_{33} and $H_{13} = H_{31}^\top$. The matrix H_{33} is a diagonal matrix containing the partial derivatives

$$\frac{\partial}{\partial d_i(z,t)}(G_{d_i}(x))_z\,, \tag{B.19}$$

where $(G_{d_i}(x))_z$ was calculated in (B.7).The columns of H_{13} can be obtained similarly by calculation of the partial derivatives of the vector representation of the gradient $G_E(x^*)$ in (B.6), i.e.

$$(H_{13})_{i_z\bullet} = \frac{\partial}{\partial d_i(z)}\,\mathrm{vec}_{\mathfrak{se}}(\mathrm{Pr}(G_E(x^*)))\,, \tag{B.20}$$

where $\bullet$ denotes the full column, and i_z denotes the index of z in a fixed ordering (e.g. column-wise). The matrix H_{11} is a bit more complicated because the *Christoffel symbols* on SE_3 are not zero. It can be calculated by the general definition of the Riemannian Hessian (cf. [1, Def. 5.5.1]). For $\eta \in \mathrm{SE}_3$ we find the following equalities

$$\begin{aligned} &\mathrm{vec}_{\mathfrak{se}}(E^{-1}\,\mathrm{Hess}_E\,\mathcal{H}(x^*,\mathbf{0},t)[E\eta]) \\ &\quad\overset{\text{Def.}}{=} \mathrm{vec}_{\mathfrak{se}}(E^{-1}\nabla_{E\eta}\mathbf{D}_E\mathcal{H}(x^*,\mathbf{0},t)) \\ &\quad\overset{\text{Lin.}}{=} \mathrm{vec}_{\mathfrak{se}}(\nabla_\eta E^{-1}\mathbf{D}_E\mathcal{H}(x^*,\mathbf{0},t)) \\ &\quad\overset{\text{(B.6)}}{=} \mathrm{vec}_{\mathfrak{se}}(\nabla_\eta G_E(x^*))\,. \end{aligned}$$

The last equation can be evaluated by using the formula in (A.10). For brevity we omit the full calculations.

Thus, evaluation of the differential equation of Z for a $\eta \in \mathfrak{g}$ and vectorization with the operator $\mathrm{vec}_{\mathfrak{g}}$ gives us the following equation for K :

$$\dot{K}(t) = -K(t)RK(t) - K(t)C(x^*,t) - C(x^*,t)^\top K(t) + H(x^*,t). \tag{B.21}$$

We want to avoid the inversion of the operator K in (B.2). Therefore we replace $P(t) := K(t)^{-1}$ in (B.21). By using the well-known rule for the calculation of the differential of an inverse matrix we obtain finally the expression in Lemma 2.

$$\begin{aligned}\dot{P}(t) =& \frac{d}{dt} K(t)^{-1} \\ =& - K(t)^{-1} \dot{K}(t) K(t)^{-1} \\ =& - P(t)(-1)K(t)RK(t)P(t) - (-1)P(t)K(t)C(x^*, t)P(t) \\ & - (-1)P(t)C(x^*, t)^\top K(t)P(t) - P(t)H(x^*, t)P(t) \\ =& R + C(x^*, t)P(t) + P(t)C(x^*, t)^\top - P(t)H(x^*, t)P(t)\end{aligned} \tag{B.22}$$

□

Proof of Theorem 1. By replacing the expression

$$Z(x^*, t)^{-1} \circ \eta = \mathrm{mat}_{\mathfrak{g}}(P(t)\,\mathrm{vec}_{\mathfrak{g}}(G(x^*, t))$$

in (B.2) we obtain the equation (15) in the paper where

$$G(x^*, t) := (G_E(x^*), \mathbf{0}_6, G_{d_i}(x^*))$$

denotes the gradient of the Hamiltonian as calculated in (B.8). The initial condition of equation (15) can be found by minimizing the value function (9) for $t = t_0$. Then we find that the optimal initial state is $x^*(t_0) = x_0$. The differential equation for P was already calculated in Lemma 2 in (B.22) and the initial state of P is the inverse of the Hessian of the value function (9) at $t = t_0$ which is R_0. This completes the proof. □

References

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